Hawking Radiation and its implications

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ABSTRACT

In this paper, I present a non-extensive summary of Hawking Radiation and current research in this field. I also include topics about the Holographic Conjecture, that entropy in the universe scales as area and not volume, and about the Information Loss Paradox about pure states being converted to mixed states when a black hole evaporates.

Introduction

In 1975, Hawking [1] showed that a black hole with surface gravity κ radiates with temperature

$$T_{\mathrm{H}} = \frac{\hbar \kappa c^3}{2\pi G k_B} \tag{1}$$

and Bekenstein [2] showed that this black hole has a corresponding entropy

$$S_{\mathrm{BH}} = \frac{c^3 A}{4\hbar G} \tag{2}$$

where A is the surface area of the event horizon.

The Four Laws of Black Hole Thermodynamics

In 1973, Bardeen, Carter and Hawking [4] published the four laws of black hole thermodynamics, listed below.

Zeroth Law. The surface gravity κ of a black hole is constant over the event horizon. This is similar to the zeroth law of ordinary thermodynamics, which states that the temperature of a sys-tem is constant when the system is in thermal equilibrium.

First Law. Two stationary black holes with small variations, δM, δJ, δQ; where M is the mass, J is the angular momentum, Q is the charge, and A is the area of the black hole,

$$\delta M = \frac{\kappa}{8\pi G c^4} \delta A + \Omega \delta J + \Phi \delta Q, \tag{3}$$

where Ω is the angular velocity of the black hole and Φ is the electric potential.

This is similar to the first law of thermodynamics

$$dU = TdS + pdV, \tag{4}$$

and the temperature T is a multiple of κ if the entropy S is a multiple of the surface area A [1]. Note that the relation (3) holds when there is no matter around the black hole, in which case the stress-energy tensor is zero. Matter terms are added to the right-hand side of (3) if there is matter around the black hole. On the horizon, the angular velocity Ω and the electric potential Φ are constant.

Second Law. The surface area A of the event horizon of a black hole never decreases, that is, the variation

$$\delta A \geq 0. \tag{5}$$

This is stronger than the second law of thermodynamics, $\delta S \geq 0$, since no black hole can split into two black holes, but two black holes can merge to form a larger black hole with surface area larger than the sum of the surface areas of the initial black holes. Thus, the surface area of each black hole cannot decrease.

Third Law. It is not possible, by a finite sequence of steps, to reduce the surface gravity κ of a black hole to zero.

This law has not been rigorously proven, but there is good reason to think this law holds, assuming weak cosmic censorship. If κ can be reduced to 0 in a finite sequence of steps, we can extend this process to yield a naked singularity, which contradicts asymptotic predictability. Cosmic censorship or asymptotic predictability is the hypothesis that singularities of gravitational collapse cannot affect events near future null infinity $\mathcal{I}^+$, and a naked singularity is a singularity without a corresponding event horizon. Thus, the weak cosmic censorship hypothesis implies that no naked singularities other than the Big Bang can exist in the universe. If this hypothesis does not hold, many results of black hole theory, including the second law (mentioned above) would fail.

The Generalized Second Law

Consider the following thought experiment proposed by Bekenstein [3]: Suppose a box with entropy $S > 0$ falls into a black hole; a black hole, to an observer outside the event horizon that is in equilibrium is uniquely characterized by the parameters M, J, and Q. The entropy of the universe outside this black hole has decreased by S, and the entropy of an object inside the black hole cannot be measured by this observer, opening the possibility that the net entropy of the universe has decreased, which is not in accordance with the second law of ordinary thermodynamics.

This is resolved by assigning an entropy $S_{\mathrm{BH}} = \eta \frac{c^3 \mathrm{A}}{\hbar G}$ to a black hole, where η is a dimensionless constant of order one and A is the surface area of the event horizon of the black hole. We then consider the *generalized second law*: The sum of the entropy of

(6)

matter outside the black hole and the entropy of the black hole S_{BH} never decreases. That is, the variation

$$\delta(S_{\text{matter}} + S_{\mathrm{BH}}) = \delta\left(S_{\text{matter}} + \eta\frac{c^3\mathrm{A}}{\hbar G}\right) \geq 0,$$

where S_{matter} is the entropy of the matter outside the black hole.

Hawking Radiation and Bekenstein-Hawking Entropy

The Hawking temperature (1) has been derived in many ways. Parikh and Wilczek present a derivation of the Hawking temperature for a Schwarzschild black hole, by determining the rate of tunneling through the black hole's event horizon [5]. Recall that the Schwarzschild metric in Painlevé-Gullstrand coordinates is given by

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)d\bar{t}^2 - \frac{2}{c}\sqrt{\frac{2GM}{c^2 r}}\,d\bar{t}dr - \frac{1}{c^2}dr^2 - \frac{r^2}{c^2}(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{7}$$

where M is the mass of the black hole. For radial null geodesics, $ds = d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2 = 0$, and dividing (7) throughout by $\frac{d\bar{t}^2}{c^2}$,

$$\left(\frac{dr}{d\bar{t}}\right)^2 + 2\sqrt{\frac{2GM}{r}}\left(\frac{dr}{d\bar{t}}\right) - \left(c^2 - \frac{2GM}{r}\right) = 0. \tag{8}$$

This is a quadratic equation in $\frac{dr}{d\bar{t}}$, hence the radial null geodesics are given by

$$\frac{dr}{d\bar{t}} = \frac{1}{2}\left(-2\sqrt{\frac{2GM}{r}} \pm \sqrt{\left(2\sqrt{\frac{2GM}{r}}\right)^2 + 4\left(c^2 - \frac{2GM}{r}\right)}\right) = -\sqrt{\frac{2GM}{r}} \pm c. \tag{9}$$

Hawking radiation is produced by the following process, as mentioned by Parikh and Wilczek [5]. A virtual pair of particles forms just inside the event horizon of a black hole, then one member of the pair tunnels outside the horizon while the other falls in. Since this is a virtual pair of particles, the particle that falls toward the singularity must be virtual and have negative energy, while the particle that tunnels out must have positive energy, for the net energy of the pair to be zero. This implies that the mass of the black hole decreases in this process. Suppose that this particle is in the s-wave (that is, in the state with the quantum number $l = 0$) with energy $\omega \ll Mc^2$, and suppose that this s-wave moves at the speed of light. The gravitational redshift of this s-wave at a distance r_0 from the singularity is $\frac{\lambda_0}{\lambda_1} = \sqrt{\left(1 - \frac{2GM}{c^2 r_0}\right)\left(1 - \frac{2GM}{c^2 r_1}\right)}$, where $r_1 = \frac{2G}{c^2}\left(M - \frac{\omega}{c^2}\right)$ is the origin of this s-wave and λ_i is the wavelength of the s-wave when measured from position r_i. When $r_i \to \infty$, the s-wave when measured from infinity is very blueshifted, since $\omega \ll Mc^2$. Thus, the Wentzel-Kramers-Brillouin (or WKB) approximation holds, that is, the functions of interest vary slowly with position. Let the generalized momentum, in the Hamiltonian formulation, be $p_i = \frac{\partial \mathcal{L}}{\partial x_i}$, where $\mathcal{L}$ is the Lagrangian. Then the action I is given by

(10)

$$I = \int_{r_0}^{r_1} p_r \, dr = \int_{r_0}^{r_1} \int_0^{p_r} d\check{p}_r \, dr,$$

where the particle travels from $r_0 = \frac{2G}{c^2}\left(M - \frac{\omega}{c^2}\right)$ (where it is produced) to $r_1 = \frac{2GM}{c^2}$, and where $\check{p}_r$ is an integration variable (the second integral on the right-hand side is a contour integral. First, Hamilton's equation is

$$\frac{dr}{d\bar{t}} = \frac{\partial \mathcal{H}}{\partial p_r} = \left.\frac{d\mathcal{H}}{dp_r}\right|_r \tag{11}$$

where $\mathcal{H}$ is the Hamiltonian. Since the particle has energy ω, this particle feels a gravitational force as if the black hole only had a mass $M - \frac{\omega}{c^2}$, since the other particle in the pair had an energy $-\omega$, hence the radial null geodesics are given by (9), replacing M by $M - \frac{\omega}{c^2}$. By (11), equation (10) becomes

$$I = \int_{r_1}^{r_0} \int_{Mc^2-\omega}^{Mc^2} \left(\frac{d\mathcal{H}}{dp_r}\right)^{-1} dr \, d\mathcal{H} = \int_{r_1}^{r_0} \int_{Mc^2-\omega}^{Mc^2} \frac{dr \, d\mathcal{H}}{{dr}/{d\bar{t}}}$$

$$= \int_{r_1}^{r_0} \int_{Mc^2-\omega}^{Mc^2} \frac{dr \, d\mathcal{H}}{c - \sqrt{\frac{2G(Mc^2 - \omega')}{c^2 r}}}\ , \tag{12}$$

because $\mathcal{H}$ is also the energy required to escape from the event horizon; when $\mathcal{H} = Mc^2 - \omega'$, the particle follows the radial null geodesics when replacing M by $M - \frac{\omega'}{c^2}$; and the mass of the black hole decreases from M to $M - \frac{\omega}{c^2}$. Note that $\frac{dr}{d\bar{t}} = c^2 - \sqrt{\frac{2GM}{r}}$ in (12) because these are outgoing geodesics; for ingoing geodesics, $\frac{dr}{d\bar{t}} = -\left(c^2 + \sqrt{\frac{2GM}{r}}\right)$. Since $\mathcal{H} = M - \frac{\omega'}{c^2}$,

$$d\mathcal{H} = -\frac{1}{c^2} d\omega', \tag{13}$$

hence by (12), the part of the action responsible for the tunneling of particles of energy ω through the event horizon is the imaginary part of the action

$$\operatorname{Im} I = \operatorname{Im}\left(\int_0^{\omega} \int_{r_0}^{r_1} \frac{-dr \, d\omega'}{c - \sqrt{\frac{2G(Mc^2 - \omega')}{c^2 r}}}\right) \tag{14}$$

where the integral over ω' is a contour integral. Then (14) implies that

$$\tag{15}$$

$$\operatorname{Im} I = \frac{4\pi G\omega}{c^5}\left(Mc^2 - \frac{\omega}{2}\right).$$

Hence the corresponding tunneling rate is given by

$$\Gamma \sim e^{-\frac{2}{\hbar}\operatorname{Im} I} = e^{-\frac{8\pi G\omega}{c^5\hbar}\left(Mc^2 - \frac{\omega}{2}\right)}. \tag{16}$$

The *Boltzmann distribution* for the distribution of particles in a system with energy ω is given by

$$F \sim e^{-\frac{\omega}{k_B T}} \tag{17}$$

where F is the distribution and T is the temperature of this system. However, the tunneling rate is the distribution, hence to lowest order in $\omega \sim \hbar$, (note ω is on the order of $\hbar$ by the Heisenberg Uncertainty Principle),

$$\frac{8\pi G\omega}{c^5\hbar}(Mc^2) = \frac{\omega}{k_B T} \Rightarrow T = \frac{c^3\hbar}{8\pi M G k_B}. \tag{18}$$

The surface gravity for a Schwarzschild black hole is $\kappa = \frac{c^2}{4GM}$, hence (18) becomes

$$T = \frac{\hbar\kappa c^3}{2\pi G k_B}, \tag{19}$$

which is the Hawking temperature in equation (1). The corresponding Bekenstein-Hawking entropy is given by

$$\Delta S_{BH} = -\frac{4c^3\pi G\omega}{\hbar}(2Mc^2 - \omega) = \frac{4c^3\pi G}{\hbar}((Mc^2 - \omega)^2 - (Mc^2)^2). \tag{20}$$

Hence,

$$S_{BH} = \frac{4\pi G}{\hbar}M^2c^7 = \frac{4\pi}{\hbar G}\left(\frac{2GM}{c^2}\right)^2\frac{\hbar c^3}{4} = \frac{c^3 A}{4\hbar G} \tag{21}$$

is the Bekenstein-Hawking entropy of a black hole whose event horizon has a surface area $A = 4\pi r_S^2 = 4\pi\left(\frac{2GM}{c^2}\right)^2$. (Note that the Schwarzschild coordinate r is defined so that the surface area of a sphere of radius r is $4\pi r^2$.)

The Holographic Conjecture

From the preceding section, the entropy of a black hole is proportional to its surface area, and not its volume. It is conjectured by Susskind [6] and ‘t Hooft [7] in 1994 that this relation holds for any gravitational system, that is, the entropy of the system is proportional to the surface area of its boundary (a two-dimensional surface) and in particular, that this property holds in the universe.

First, the planck length $l = \sqrt{\hbar G / c^3}$ is the smallest measurable scale, hence a minimum of the cube of the planck length of space is required to store a bit of information. The number of sequences of bits that can be stored in a space $\mathfrak{M}$ with volume $\mathcal{V}$ is $2^{\mathcal{V}/l^3}$ where l is the Planck length. The corresponding entropy of this system, given three degrees of freedom, is thus

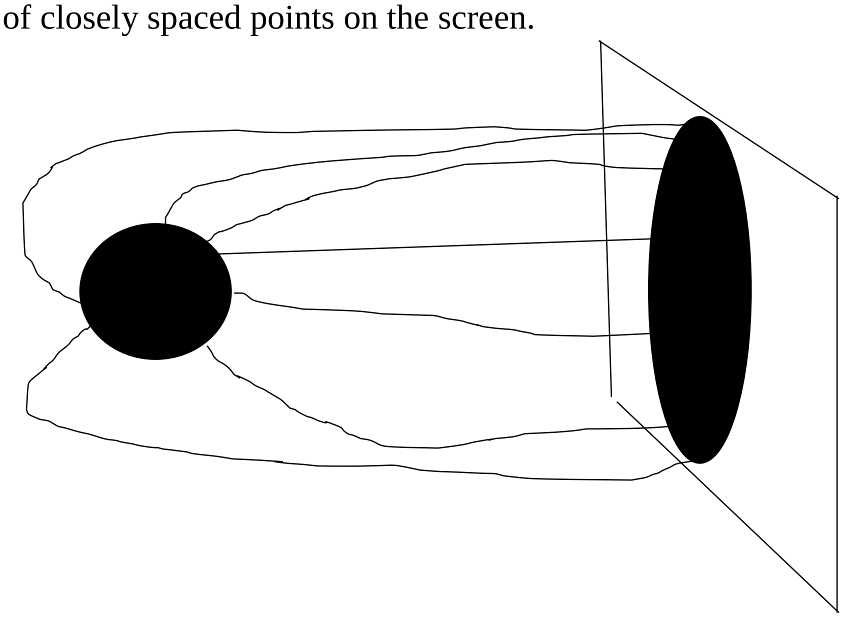
$$S_{\mathfrak{M}} = \ln\left(2^{\mathcal{V}/l^3}\right) = \frac{\mathcal{V}}{l^3}\ln 2 \sim \frac{\mathcal{V}}{l^3}. \tag{22}$$

The entropy of a black hole $\mathfrak{B}$ occupying the space $\mathfrak{M}$ (we assume that $\mathfrak{M}$ is roughly spherical) must be less than the entropy of some region of space $\mathfrak{D} \subsetneq \mathfrak{M}$. Enough matter can be added to this region $\mathfrak{D}$ to collapse it into a black hole. The generalized second law of thermodynamics states that this new black hole must have a higher entropy than that of the state that formed it. However, this black hole is contained in $\mathfrak{D}$ and is hence strictly contained in $\mathfrak{M}$. However, such a black hole must have less entropy than a black hole occupying the space $\mathfrak{M}$, contradicting the preceding assumptions. This argument shows that this region $\mathfrak{M}$ must only have an entropy proportional to its surface area (as do black holes) and not to its volume.

Susskind [6] has proposed that no bit of information on a screen can be stored in less than one planck area, or square planck length

$$l^2 = \frac{\hbar G}{c^3}. \tag{23}$$

Figure 1. The projection of a black hole onto the screen. Note that this projection will appear as a series of closely spaced points on the screen.



A *parton* is defined an elementary constituent of matter. Consider a two-dimensional screen, the image of a black hole $\mathfrak{B}$, whose event horizon has a surface area A, is the set of all points on the screen such that when a light ray is emitted perpendicular to the screen, then the corresponding null geodesic ends in $\mathfrak{B}$. (See Figure 1.) Then if A_{screen} is the area of the projection of the black hole on the screen, then

$$A_{\text{screen}} \sim A. \tag{24}$$

Now the *focusing theorem* states that if a packet of light has area $a(\lambda)$ (λ is the affine parameter for null geodesics) then the derivative

$$\frac{d^2}{d\lambda^2}\left(\sqrt{a}\right) = \frac{1}{2\sqrt{a}}\frac{da}{d\lambda} \leq 0. \tag{25}$$

Then the area a of this packet of light decreases (not necessarily strictly) with time, hence any area element dA on the surface of the event horizon of $\mathfrak{B}$ is mapped to a larger area element

$$dA_{\text{screen}} \geq dA. \tag{26}$$

Since the event horizon has an information density of one bit per planck area, the information density on the screen is at least one bit per planck area.

Suppose that another black hole $\mathfrak{B}'$ is behind $\mathfrak{B}$. It appears initially as if no null geodesic corresponding to a photon being emitted perpendicular to the screen ends in the black hole $\mathfrak{B}'$. However, the black hole $\mathfrak{B}$ acts as a gravitational lens, so that when $\mathfrak{B}'$ is directly behind $\mathfrak{B}$, $\mathfrak{B}'$ forms an Einstein ring around $\mathfrak{B}$, and an application of the focusing theorem (as above) shows that the information density does not decrease. Since the black hole is the most compact object, this reasoning holds for any spaces $\mathfrak{N}$ and $\mathfrak{N}'$.

This reasoning, though, is only semi-classical, and more quantum mechanics is required to give the true picture.

The Information Loss Paradox

Recall from (1) that the Hawking temperature T_H is inversely proportional to the square root of the surface area of the hole, and by the Stefan-Boltzmann Law, the rate of loss of mass is

$$\frac{dM}{dt} = -\epsilon\sigma T_H^4 A, \tag{27}$$

where A is the area of the event horizon of the black hole and ϵ is the *emissivity* (this arises because black holes are not blackbodies), as mentioned in a review by Prof. Carlip [8]. Then the rate of change of mass is inversely proportional to the surface area of the black hole, that is, the rate of change of mass is inversely proportional to the square of the mass of the hole (since the surface area of a black hole is proportional to the square of its radius, which is proportional to the square of its mass). Thus, in a finite amount of time, the black hole will evaporate.

When a unitary operator U (that is, $U^\dagger U = 1$) acts on a pure state $|\psi\rangle$, then the new state $U|\psi\rangle$ remains pure. However, if the Hawking radiation is thermal radiation and is in a mixed state, and the evaporation process of black hole formed from a pure state is unitary, then a unitary operator changes a pure state which formed the black hole to the mixed state, which is Hawking radiation. This is known as the *information loss paradox*.

This paradox is a topic of intense debate. According to Hawking [9], there is no information loss resulting from this procedure. Recall that the outgoing Eddington-Finkelstein coordinate v (also called the advanced coordinate) is given by

$$v = t + r_* \tag{28}$$

where r_* is the tortoise coordinate. Then Hawking [9] says that the information on the ingoing particle is stored on the horizon as well as a translation of the Eddington-Finkelstein coordinate v, in which case the Hawking radiation is actually a pure state. It also turns out that this construction yields unitary evolution.

Another theory, proposed by Mathur [10] is that there are no true black holes. Instead, they are *fuzzballs*, that is, they have an alternative geometry that includes an event horizon, but there is no internal singularity, instead the information is distributed roughly uniformly in the fuzzball's interior. However, this requires that string theory holds.

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